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Majorization for Products of Measurable Operators

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A new equality for a faithful normal semifinite trace on a von Neumann algebra is proved. We conjecture a strengthening of the result.

Let M be a semifinite von Neumann algebra acting in a Hilbert space H, let μ be a faithful normal semifinite trace on M, let $P(M)$ be the lattice of all projections in M. A closed operator x affiliated with M having everywhere dense domain of definition $D(x)$ is called μ-measurable (Nelson, 1974) if for every $\varepsilon > 0$ there exists $q \in P(M)$ such that $q(H) \subset D(x)$ and $\mu(q^{\perp}) < \varepsilon$, where $q^{\perp} = e - q$, e is the unit in M. The set $K(M, \mu)$ of all μ-measurable operators is a *-algebra with respect to the strong sum, the strong product, and the adjoint operator. For any subset $L \subset K(M, \mu)$ we shall denote by L^+ the set of all positive self-adjoint elements from L. The rearrangement $\mu_t(x)$ of an operator $x \in K(M, \mu)$ is the function defined by

$$\mu_t(x) = \inf\{\|xq\|_M: q \in P(M), \qquad \mu(q^{\perp}) \leq t\}, \qquad t > 0$$

where $\|\cdot\|_M$ is the C^*-norm on M. The set $K_0(M, \mu) = \{x \in K(M, \mu): \lim_{t\to+\infty} \mu_t(x) = 0\}$ is a *-subalgebra and two-sided ideal in $K(M, \mu)$. For $1 \leq p < +\infty$, let L_p denote the noncommutative Lebesgue spaces associated with (M, μ) (Yeadon, 1975). We denote by μ the extension of μ from $(M \cap L_1)^+$ to a unique bounded functional on $M \cap L_1$, and then over the whole L_1.

Lemma 1 (Stroh and West, 1993). $K(M, \mu) = K_0(M, \mu) + M$, $K(M, \mu)^+ = K_0(M, \mu)^+ + M^+$ [i.e., every $y \in K(M, \mu)$ is of the form $y_1 + y_2$ with $y_1 \in K_0(M, \mu)$, $y_2 \in M$].

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Lemma 2 (Brown and Kosaki, 1990). If $a, b \in K(M, \mu)$ and $ab, ba \in L_1$, then $\mu(ab) = \mu(ba)$.

Lemma 3. $\mu_t(z^*z) = \mu_t(zz^*)$ for every $z \in K$. In particular,

$$z^*z \in L_1 \Leftrightarrow zz^* \in L_1 \Rightarrow \mu(z^*z) = \mu(zz^*)$$

Theorem 1. Let $x, y \in K(M, \mu)^+$ satisfy $xy \in L_1$. Then

$$x^{1/2}yx^{1/2},\ y^{1/2}xy^{1/2} \in L_1^+ \qquad \text{and} \qquad \mu(xy) = \mu(x^{1/2}yx^{1/2}) = \mu(y^{1/2}xy^{1/2})$$

Proof. 1. Let $x, y \in M^+$. Without loss of generality we suppose that $\|y\|_M \leq 1$. For every $n \in \mathbb{N}$, introduce a continuous function on $[0, +\infty)$ as follows: $f_n(t) = t(t + n^{-1})^{-1/2}$, if $0 \leq t \leq 1$, f_n is linear on $[1, 2]$ with $f_n(2) = 1$, and $f_n(t) = 1$ for $t > 2$. Obviously $0 \leq f_n \leq 1$. Consider the operators $y_n \equiv (y + n^{-1}e)^{-1/2} \in M^+$ and $z_n \equiv f_n(y) \cdot xy \cdot y_n \in L_1$. Actually, we have $z_n = f_n(y)\ x \cdot f_n \in L_1^+$, $n \in \mathbb{N}$. As the function sequence (f_n) uniformly converges to

$$f(t) = \begin{cases} \sqrt{t} & \text{if} \quad 0 \leq t \leq 1 \\ 1 & \text{if} \quad t > 1 \end{cases}$$

on $[0, +\infty)$, it follows that, by (b) in Theorem VII.2 of Reed and Simon (1972), we obtain $\|f_n(y) - y^{1/2}\|_M \to 0$, $n \to \infty$. Thus $z_n \to y^{1/2}xy^{1/2} (n \to \infty)$ in the ultraweak topology. As a normal trace is ultraweakly lower semicontinuous (Dixmier, 1969, p. 85), we have

$$\begin{aligned} &\mu(y^{1/2}xy^{1/2}) \\ &\leq \liminf_{n \to +\infty} \mu(z_n) = \liminf_{n \to +\infty} \mu(f_n(y) \cdot xy \cdot y_n) \\ &= \liminf_{n \to +\infty} \mu(xy \cdot y_n \cdot f_n(y)) \leq \liminf_{n \to +\infty} \|y_n \cdot f_n(y)\|_M \cdot \mu(|xy|) \\ &\leq \mu(|xy|) < +\infty \end{aligned}$$

[On the third line, we have gone over to the fourth by making use of Lemma 2 with $a = f_n(y)$, $b = xy \cdot y_n$ and of the inequality $f_n(t)(t + n^{-1})^{-1/2} < 1$ for all $0 \leq t \leq 1$]. Thus $y^{1/2}xy^{1/2} \in L_1^+$. By Lemma 2 with $a = y^{1/2}$, $b = xy^{1/2}$, one has $\mu(xy) = \mu(y^{1/2}xy^{1/2})$. It remains to apply Lemma 3 with $z = y^{1/2}x^{1/2}$.

2. The special case $y \in K_0(M, \mu)^+$ has been examined in Theorem 3.4 of Dodds *et al.* (1993).

3. Consider the case when $y \in K(M, \mu)^+$, $x \in M^+$. Let $y = y_1 + y_2$ be the representation as in Lemma 1. For $\alpha = \lim_{t \to +\infty} \mu_t(y)$ and the sets $A = [0, \alpha]$ and $B = (\alpha, +\infty)$ it holds that $y_2 = y \cdot e_A(y) + \alpha \cdot e_B(y)$ (Stroh and West, 1993), wherein $e_T(y)$ is the spectral projection of y corresponding to the Borel set $T \subset \mathbb{R}$. Thus

$$x \cdot y_2 = \mathrm{xy} \cdot e_A(y) + \alpha x \cdot e_B(y) \tag{1}$$

and the first summand in the right-hand side belongs to L_1. By the "commutative" inequality $\alpha^2 e_B(y) \leq y^2$, the square of the absolute value of the adjunction of the second summand can be estimated as $|\alpha e_B(y)x|^2 = x \cdot \alpha^2 e_B(y) \cdot x \leq xy \cdot yx$. The operator monotonicity of the function $t \mapsto \sqrt{t}$ $(t \geq 0)$ entails $|\alpha e_B(y)x| \leq |yx|$. Therefore, the second summand in (1) belongs to L_1, too. Hence $xy_2 \in L_1$. Now, we have $xy_1 = xy - xy_2 \in L_1$. By applying point 2 of the proof to x, y_1 and point 1 to x, y_2, we have $x^{1/2}y_1x^{1/2}y_2x^{1/2} \in L_1$. Thus $x^{1/2}yx^{1/2} \in L_1{}^+$. It remains to repeat the concluding arguments in the proof of point 1.

4. The general case: $x, y \in K(M, \mu)^+$. Let $x = x_1 + x_2$ be the representation as in Lemma 1. By the arguments of point 3, we obtain x_1y, $x_2y \in L_1$. By applying points 2 and 3 of the proof to the couples x_1, y and x_2, y, respectively, we have $y^{1/2}x_1y^{1/2}$, $y^{1/2}x_2y^{1/2} \in L_1{}^+$. Thus $y^{1/2}xy^{1/2} \in L_1{}^+$. Now, we apply Lemma 3 with $z = y^{1/2}x^{1/2}$. QED

In what follows, $z^n \equiv z \cdot \ldots \cdot z$ is the product of n copies of $z \in K(M, \mu)$.

Corollary 1. Let $p \in \mathbb{N}$ and $x, y \in K(M, \mu)^+$ be such that $(xy)^p \in L_1$. Then $x^{1/2}yx^{1/2}, y^{1/2}xy^{1/2} \in L_p{}^+$, and $\mu((xy)^p) = \mu((x^{1/2}yx^{1/2})^p) = \mu((y^{1/2}xy^{1/2})^p)$.

Proof. We apply Theorem 1 to x, $y_1 \equiv y \cdot (xy)^{p-1} \in K(M, \mu)^+$. Observe that $x^{1/2}y_1x^{1/2} = (x^{1/2}yx^{1/2})^p$. Since $((xy)^p)^* = (yx)^p \in L_1$, it analogously follows that $\mu((y^{1/2}xy^{1/2})^p) = \mu((yx)^p)$. Lemma 2 yields

$$\mu((xy)^p) = \mu(xy_1) = \mu(y_1x) = \mu((yx)^p)$$

If $xy \in L_p$, then $(xy)^p \in L_1$. QED

Corollary 2. Let $\mu(e) < +\infty$ and $x, y \in K(M, \mu)^+$ satisfy $xy \in M$. Then $x^{1/2}yx^{1/2}$, $y^{1/2}xy^{1/2} \in M^+$.

Proof. Without loss of generality we suppose that $\mu(e) = 1 = \|xy\|_M$. Then $xy \in L_p$ and $\|xy\|_p = (\int_0^1 \mu_t(xy)^p dt)^{1/p} \leq 1$, $1 \leq p < \infty$. For every $p \in \mathbb{N}$,

$$\|x^{1/2}yx^{1/2}\|_p = \mu((x^{1/2}yx^{1/2})^p)^{1/p} = \tau((xy)^p)^{1/p} \leq 1$$

It is easy to show that $\lim_{p\to+\infty} \|x^{1/2}yx^{1/2}\|_p = \|x^{1/2}yx^{1/2}\|_M \leq 1$. Similarly, $\|y^{1/2}xy^{1/2}\|_M \leq 1$. QED

For $x, y \in K(M, \mu)$, the *submajorization* (= the weak spectral order of Hardy, Littlewood, and Polya) $y \prec x$ means that $\int_0^t \mu_s(y)\, ds \leq \int_0^t \mu_s(x)\, ds$, $\forall t > 0$.

Conjecture A. If $x, y \in K(M, \mu)^+$, then $x^{1/2}yx^{1/2} \prec xy$.

This conjecture strengthens Theorem 1.

Let us prove that if Conjecture A holds for every continuous semifinite von Neumann algebra, then it is valid for all semifinite von Neumann algebras.

Consider the commutative W^*-algebra $N = L_\infty(0, 1)$ with the trace $\nu(f) = \int_0^1 f\, dm$, where m is the Lebesgue measure on [0,1], and regard N as acting in $F = L_2(0, 1)$. Let $L = M \otimes N$ be the tensor product of von Neumann algebras M and N, and let $\lambda = \mu \otimes \nu$ be the tensor product of the traces μ and ν. It is clear that the algebra L has no atoms. Let $x \in K(M, \mu)$ and D be the linear subspace in $H \otimes F$ generated by the vectors of the form $\xi \otimes \eta$, $\xi \in D(x)$, $\eta \in F$. For every $\xi = \Sigma_{i=1}^n \xi_i \otimes \eta_i$ we put $(x \otimes e)(\xi) = \Sigma_{i=1}^n x\xi_i \otimes \eta_i$. The linear operator $x \otimes e$ with the domain of definition D is preclosed and its closure $x \overline{\otimes} e$ belongs to $K(L, \lambda)$ (Stinespring, 1959). Thus, $K(M, \mu) \overline{\otimes} e = \{x \overline{\otimes} e: x \in K(M, \mu)\}$ is a *-subalgebra in $K(L, \lambda)$ and $\mu_t(x) = \lambda_t(x \overline{\otimes} e)$ for all $x \in K(M, \mu)$, $t > 0$, where $\lambda_t(x \overline{\otimes} e)$ is the rearrangement calculated with respect to the trace λ of the operator $x \overline{\otimes} e$. We have

$$\begin{aligned}
&\int_0^t \mu_s(x^{1/2}yx^{1/2})\, ds \\
&= \int_0^t \lambda_s(x^{1/2}yx^{1/2} \overline{\otimes} e)\, ds = \int_0^t \lambda_s((x^{1/2} \overline{\otimes} e)(y \overline{\otimes} e)(x^{1/2} \overline{\otimes} e))\, ds \\
&= \int_0^t \lambda_s((x \overline{\otimes} e)^{1/2}(y \overline{\otimes} e)(x \overline{\otimes} e)^{1/2})\, ds \le \int_0^t \lambda_s((x \overline{\otimes} e)(y \overline{\otimes} e)\, ds \\
&= \int_0^t \lambda_s((xy \overline{\otimes} e)\, ds = \int_0^t \mu_s(xy)\, ds \quad \text{QED}
\end{aligned}$$

Let $\mathscr{L} = \mathscr{L}(H)$ denote the algebra of all bounded linear operators on H. We denote by Tr the canonical trace on $\mathscr{L}$ and let $\sigma(z)$ be the spectrum of $z \in \mathscr{L}$. For $x \in \mathscr{L}$ let $s(x)$ denote the sequence of the s-numbers of x. Then $K(\mathscr{L}, \mathrm{Tr}) = \mathscr{L}$ and $\mu_t(x) = \Sigma_{i=1}^\infty s_n(x)\lambda\, [n-1, n)\, (t)$ $K_0\, (\mathscr{L}, \mathrm{Tr})$ is the algebra of all compact operators on H. For $x \in K_0(\mathscr{L}, \mathrm{Tr})$ let $\lambda(x)$ denote the nonincreasing sequence of the eigenvalues of x with regard to their multiplicity. Then $s(x) = \lambda((x^*-x)^{1/2})$. For $x, y \in \mathscr{L}$ the submajorization $y \prec x$ means that $\Sigma_{n=1}^k s_n(y) \le \Sigma_{n=1}^k s_n(x)$ for all $k \in \mathrm{N}$.

Theorem 2. If $x, y \in \mathscr{L}^+$ and $xy \in K_0\, (\mathscr{L}, \mathrm{Tr})$, then $x^{1/2}yx^{1/2} \prec xy$

Proof. For every $z, t \in \mathscr{L}$ we have $\sigma(zt)\setminus\{0\} = \sigma(tz)\setminus\{0\}$, and if some $\lambda \neq 0$ is an eigenvalue of zt, then it is an eigenvalue of tz with the same multiplicity (Reed and Simon, 1978). It is easy to show that

$$xy \in K_0(\mathscr{L}, \mathrm{Tr}) \Leftrightarrow x^{1/2}xy^{1/2} \in K_0(\mathscr{L}, \mathrm{Tr})^+$$

As $xy = x^{1/2}\,(x^{1/2}y)$, it follows that $\lambda_n(x^{1/2}yx^{1/2}) = \lambda_n(xy)$ for all $n \in \mathrm{N}$. Since $x^{1/2}yx^{1/2}$ is self-adjoint and positive, we have $\lambda_n(x^{1/2}yx^{1/2}) = s_n(x^{1/2}yx^{1/2})$.

Now, by the Corollary from the Weil Majorization Theorem (Gohberg and Krein, 1965, p. 62) we obtain

$$\sum_{n=1}^{k} s_n(x^{1/2}yx^{1/2}) = \sum_{n=1}^{k} \lambda_n(x^{1/2}yx^{1/2}) = \sum_{n=1}^{k} \lambda_n(xy) \leq \sum_{n=1}^{k} s_n(xy) \quad \text{QED}$$

Recall that a linear subspace E of K_0 ($\mathcal{L}$, Tr) endowed with a norm $\|\cdot\|_E$ with respect to which E is a Banach space is called *a symmetric space* provided that $x \in E$, $y \in \mathcal{L}$, and $s(y) \leq s(x)$ imply $y \in E$ and $\|y\|_E \leq \|x\|_E$.

A symmetric space E is called *fully symmetric*, if from $y \prec x, x \in E$ and $y \in \mathcal{L}$ imply that $y \in E$, and $\|y\|_E \leq \|x\|_E$.

The following corollary extends a result for the classical von Neumann–Shatten ideals $\mathcal{S}_p$, $1 \leq p < +\infty$, in Reed and Simon (1978).

Corollary 3. If E is a fully symmetric space and $x,y \in \mathcal{L}^+$ and $xy \in E$, then $x^{1/2}yx^{1/2} \in E^+$ and $\|x^{1/2}yx^{1/2}\|_E \leq \|xy\|_E$.

The following assertion generalizes the Golden–Thompson–Ruskai inequality (Ruskai, 1972):

Corollary 4. Let x, y be self-adjoint operators on H, bounded above, and $x + y$ is essentially self-adjoint. If $e^x e^y \in \mathcal{S}_1$, then $\mathrm{Tr}(e^{x+y}) \leq \mathrm{Tr}(e^x e^y) = \mathrm{Tr}(e^{x/2} e^y e^{x/2})$.

Remarks. 1. For every symmetric space $E \neq K_0$ ($\mathcal{L}$, Tr) there exist $x,y \in P(\mathcal{L})$ such that $x^{1/2}yx^{1/2} = xyx \in E^+$, $xy \notin E$. This results from Loebl (1986): $E \neq K_0$ ($\mathcal{L}$, Tr) $\Leftrightarrow \exists x \in E^+$: $x^{1/2} \notin E^+$, and from the known fact that every contraction $x \in \mathcal{L}^+$ with dimKer $x = \infty$ is of the form $x = rqr$ for suitable $r, q \in P(\mathcal{L})$.

2. Theorem 2 is best possible in the sense that $s(x^{1/2}yx^{1/2}) \leq s(xy)$ fails to be true in general. For example, consider in $M_2(\mathbb{C})$

$$x = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \qquad y = \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}$$

Then we have

$$x^{1/2}yx^{1/2} = \frac{1}{5}\begin{pmatrix} 7 & 6 \\ 6 & 23 \end{pmatrix}, \qquad s_1(x^{1/2}yx^{1/2}) = 5, \qquad s_2(x^{1/2}yx^{1/2}) = 1$$

For xy, $s_1(xy) = (15 + 10\sqrt{2})^{1/2} > 5$ and $s_2(xy) = (15 - 10\sqrt{2})^{1/2} < 1$.

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